Optimal Lift Control of a Hypersonic Lifting Body during Atmospheric Entry

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The problem of optimal lift control of a hypersonic lifting body during planetary entry was considered. Variation of atmospheric density was assumed to be exponential. The case treated is a generalization of an earlier investigation in which a constant atmospheric density model was used. It was shown that the optimal path consists of two kinds of subarcs, i.e., extremal lift subarcs and singular subarcs. Analytic solutions for state variables and the Lagrange multipliers were obtained in closed form for both subarcs. It was concluded that the lift on the singular subarc is always dominated by the term due to the variation of the atmospheric density. Furthermore, boundedness of the control renders the singular subarc inadmissible above a certain altitude that depends on the value of the initial re-entry angle and the initial altitude. The usual questions concerning the number of subarcs, their sequence, and the location of switching points were fully answered for two examples and partially answered for a third example considered here.

Nomenclature

B = mg/C_DS
C_D = drag coefficient
C_L = $C_L(\alpha)$ = lift coefficient
D = drag = $mK_DV^2e^{-\beta h}$
g = gravitational acceleration at $h = 0$
h = altitude
H = Hamiltonian
K_D = $\frac{1}{2}\rho_0SC_D/m$
K_L = $\frac{1}{2}\rho_0SC_L/m$
K_{LM} = maximum (K_L)
L = lift = $mK_LV^2e^{-\beta h}$
m = mass of the re-entry vehicle
R = mean earth radius
S = effective cross section of vehicle
s = $R\sigma$ = range
V = velocity
α = control variable or angle of attack
β = $900/R$ for the earth
γ = flight-path angle is measured counterclockwise from the local horizon
$\lambda_h,\lambda_V,\lambda_\gamma,\lambda_s$ = Lagrange multipliers
ρ = $\rho_0\exp(-\beta h)$ = atmospheric density
ρ_0 = atmospheric density at $h = 0$
σ = angle between reference axis and vehicle

Subscripts

f = terminal conditions
i = initial conditions or conditions at the beginning of each subarc

I. Introduction

ANALYTIC solutions of lifting re-entry into a planetary atmosphere have been discussed extensively in Refs. 1–4. Most of these solutions were derived either under the assumption of constant lift-to-drag ratio or both constant lift and constant drag coefficients.

Owing to the difficulty of obtaining analytic solutions to problems of optimal lift control of gliding vehicles, most of the available solutions for optimal lift control during re-entry are numerical. A class of approximate analytic solutions was considered by Miele[5,6] who treated optimization of the lift program of a gliding vehicle under the assumption of a shallow, smooth glide path. Recently, Pottsepp and Shi[7] showed that an additional subclass of optimum trajectory problems for hypersonic gliding vehicles may be solved analytically under the assumption of constant density, gravity, and drag coefficient.

Analytic solutions of simplified optimal re-entry problems are important from the point of view of serving as a basis for investigation of more complicated cases, as well as providing a general understanding of optimal use of lift controls. Analytic solutions also provide a better foundation for the solution of guidance problems as typical guidance laws are based on such relatively restrictive assumptions as either constant drag, constant altitude or equilibrium glides as discussed in Refs. 8 and 9.

In this paper, the optimal lift control of a re-entry vehicle is investigated analytically by retaining the concept of a constant drag coefficient used in an earlier investigation (Ref. 7), but relaxing the assumption of constant density. This also corresponds to retaining all the assumptions made in studying atmospheric entry problems in Refs. 1, 2, 3, and 4, except that the assumption of constant lift coefficient is relaxed to allow the lift to be used as a control. The maximum value of the lift-to-drag ratio is assumed to be 0.5 or greater as recently discussed by Tannas (Refs. 8 and 9). The present solution is believed to provide some general ideas on optimal lift control applications and will serve as a first step in solving the more realistic case of variable drag coefficient which is to be reported in a later publication.

II. Governing Equations

The governing equations (cf. Ref. 10) for a gliding re-entry vehicle can be written in the form

$$\dot{V} = -D/m - g[R^2/(R+h)^2]\sin\gamma \tag{1}$$

$$V(\dot{\gamma} - \dot{\sigma}) = L/m - g[R^2/(R+h)^2]\cos\gamma \tag{2}$$

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$$\dot{h} = V \sin\gamma \tag{3}$$

$$\dot{\sigma} = V \cos\gamma/(R + h) \tag{4}$$

If $h/R \ll 1$ and $s = R\sigma$ are introduced, the governing equations can be simplified to those of Refs. 3 and 4;

$$\dot{V} = -D/m - g \sin\gamma \tag{5}$$

$$\dot{\gamma} = L/mV - (g - V^2/R) \cos\gamma/V \tag{6}$$

$$\dot{h} = V \sin\gamma \tag{7}$$

$$\dot{s} = V \cos\gamma \tag{8}$$

As discussed in Refs. 1 and 3, the isothermal atmosphere has long been adopted as a first approximation for studies of planetary entry. Hence

$$d\rho/dh = -\beta\rho \tag{9}$$

where β is a constant and

$$\beta R = 900 \tag{10}$$

for the earth and R is the mean radius of the earth.

In several publications (cf. Refs. 1–4) on the subject of re-entry, the assumption

$$D/m \gg g \sin\gamma \tag{11}$$

was adopted.

In general, this assumption does not hold in the upper section of a trajectory starting at a high altitude. In order to include the higher altitude portion of a trajectory, the problem may be separated into two regions:

I. $$D/m \gg g \sin\gamma \tag{12}$$

II. $$D/m \ll g \sin\gamma \tag{13}$$

However, typically, previous investigators adopted the condition (12) for the entire trajectory and neglected the gravitational force in the velocity equation. The justifications for such an assumption have been given previously by Allen and Eggers,[1] and Citron and Meir[4] and were illustrated by numerical examples. The validity of the assumption lies in the fact that in the upper portion of the re-entry trajectory, the velocity is high and is not affected significantly by either the drag or the gravity, both of which are of higher order. In the lower portion, however, the drag is dominant and the gravity is of higher order.

With the preceding assumptions, the variational Hamiltonian H for the system of equations of state variables and Lagrange multipliers (i.e., λ_V, λ_γ, λ_h, and λ_s) can be written as follows:

$$H = -\lambda_V \frac{D}{m} + \lambda_\gamma \left[\frac{L}{mV} - \left(g - \frac{V^2}{R}\right)\frac{\cos\gamma}{V}\right] + \lambda_h V \sin\gamma + \lambda_s V \cos\gamma \tag{14}$$

The optimality condition requires

$$\partial H/\partial\alpha = \lambda_\gamma/mV\, \partial L/\partial\alpha = 0 \tag{15}$$

with

$$L = mK_L V^2 e^{-\beta h} = \tfrac{1}{2}\rho_0 V^2 S C_L(\alpha) e^{-\beta h} \tag{16}$$

where ρ_0 and S are constants. For typical hypersonic vehicles, the function $C_L(\alpha)$ is periodic with the period 2π, possesses a single maximum and minimum, and has no horizontal inflection points in the interval $-\pi/2 \leq \alpha \leq \pi/2$ (Ref. 7). Since the problem is linear in $C_L(\alpha)$ and nonlinear in the state variables, the existence of singular subarcs may be expected.

Equation (15) can be satisfied by either

$$\lambda_\gamma = 0 \text{ and } L_{\min} \leq L \leq L_{\max} \tag{17}$$

or

$$\lambda_\gamma \neq 0 \text{ and } L = L_{\max} \text{ or } L_{\min} \tag{18}$$

For a given vehicle, the upper bound of the control is given as

$$C_L(\alpha) = C_{L_{\max}} \text{ or } L_{\max} = mK_{LM}V^2 e^{-\beta h} \tag{19}$$

For the purpose of simplifying the subsequent discussions, we assume that the lower bound

$$C_{L_{\min}} = -C_{L_{\max}} \text{ or } L_{\min} = -L_{\max} \tag{20}$$

Thus, the solutions in the present problem consist of two kinds of subarcs; the singular subarc, as defined by Eq. (17), and the nonsingular subarcs (i.e., maximum lift subarc and minimum lift subarc), defined by Eq. (18).

In the interest of simplifying the discussion, the following analysis is restricted to the case where

$$H \equiv 0 \tag{21}$$

The problems of achieving maximum terminal velocity (i.e., minimum energy loss due to aerodynamic drag) and maximum final range fall into this category. Fortunately, analytic solutions for these cases can be obtained in closed form for both the state variables and the multipliers.

III. Solutions on the Singular Subarc

Observing Eqs. (14) and (17), the Euler-Lagrange equations can be written as follows:

$$\dot{\lambda}_V = \lambda_V 2D/mV - \lambda_h \sin\gamma - \lambda_s \cos\gamma \tag{22}$$

$$\dot{\lambda}_\gamma = -\lambda_h V \cos\gamma + \lambda_s V \sin\gamma = 0 \tag{23}$$

$$\dot{\lambda}_h = -\lambda_V\, \beta D/m \tag{24}$$

$$\dot{\lambda}_s = 0 \tag{25}$$

Differentiating Eq. (23),

$$\ddot{\lambda}_\gamma = -\dot{\lambda}_h \cos\gamma + (\lambda_h \sin\gamma + \lambda_s \cos\gamma)\dot{\gamma} = 0 \tag{26}$$

Hence, the lift control on the singular subarc can be calculated from Eqs. (14, 21, 23, 25, and 26) as

$$\dot{\gamma} = -\beta V \cos\gamma \tag{27}$$

or

$$L_s = -m\beta V^2 \cos\gamma(1 + g/\beta V^2 - 1/\beta R) \tag{28}$$

It can be easily shown that the first term in Eq. (28) is always dominant. If the terminal velocity is above 2000 fps, $g/\beta V^2 \leq 0.018$, considering the earth's gravity and atmosphere. Another distinct feature of the singular subarc is that the absolute value of the lift needed for optimal control is greater than the maximum lift $L_{\max}$ available above a certain altitude which depends on the initial flight-path angle γ_i and the initial altitude h_i. Thus, the low density of the air at high altitude precludes the use of the singular subarc because it is impossible to obtain the required lift except on an almost vertical singular subarc. Switch to the singular subarc is possible only if

$$|\beta \cos\gamma(1 + g/\beta V^2 - 1/\beta R)| \leq K_{LM}e^{-\beta h} \tag{29}$$

The point where the equal sign applies will be called the critical point (CP). The condition (29) must be satisfied on the entire singular subarc. For this reason, most optimum re-entry paths will start with a nonsingular subarc (i.e., maximum or minimum lift subarc). A detailed discussion on the bounded control was also given in Ref. 11 by Leitmann.

With the lift given by Eq. (28) and the assumption adopted in Eq. (12), Eqs. (5) and (6) become

$$\dot{V} = -D/m \tag{30}$$

and

$$\dot{\gamma} = -\beta V \cos\gamma \tag{31}$$

respectively.

We can now proceed to obtain solutions for the state variables. From Eqs. (8) and (31),

$$d\gamma/ds = -\beta \tag{32}$$

or

$$\gamma - \gamma_i = -\beta(s - s_i) \tag{33}$$

where the subscript i denotes the initial conditions at the beginning of each subarc. The preceding solution shows that the flight-path angle γ decreases as s increases.

Equations (7, 8, and 33) give

$$dh/ds = \tan[\gamma_i - \beta(s - s_i)] \tag{34a}$$

and

$$dh/d\gamma = -(1/\beta)\tan\gamma \tag{34b}$$

or

$$h - h_i = \frac{1}{\beta}\ln\frac{|\cos[\gamma_i - \beta(s - s_i)]|}{|\cos\gamma_i|} \tag{35a}$$

and

$$h - h_i = \frac{1}{\beta}\ln\frac{|\cos\gamma|}{|\cos\gamma_i|} \tag{35b}$$

Similarly, Eqs. (7, 9, and 31) provide

$$\rho = \rho_i|\cos\gamma_i|/|\cos\gamma| \tag{36}$$

The velocity can be determined by directly integrating Eq. (30);

$$\ln\frac{V}{V_i} = \frac{K_D\rho_i\cos\gamma_i}{\beta\rho_0}(\tan\gamma - \tan\gamma_i) \tag{37}$$

It may be seen from Eq. (35) that for a given initial point (s_i,h_i), the shape of the singular arc depends only on the initial value of the flight-path angle γ_i. Hence, it represents a family of geometric curves with a parameter γ_i as shown in Fig. 1 for $h_i = 130{,}000$ ft and $s_i = 100{,}000$ ft. It is also pointed out that some of these solutions become undefined with respect to γ at $\gamma_i = -\pi/2$. In this limiting case, γ is not a suitable independent variable. However, V and h remain suitable variables and their relation can be obtained from Eqs. (7) and (30) as follows:

$$\ln(V/V_i) = (K_D/\beta)(e^{-\beta h} - e^{-\beta h_i}) \tag{38}$$

Equation (38) can also be obtained by eliminating γ from Eqs. (36) and (37) as γ becomes $-\pi/2$. The corresponding Euler-Lagrange equations for this problem can be written as

$$\dot{\lambda}_V = \lambda_V\frac{D}{mV} \tag{39}$$

$$\dot{\lambda}_\gamma = -\lambda_h V\cos\gamma + \lambda_s V\sin\gamma = 0 \tag{40}$$

$$\dot{\lambda}_h = -\lambda_V\,\beta D/m \tag{41}$$

$$\dot{\lambda}_s = 0 \tag{42}$$

The solutions are given below without a detailed derivation.

$$\lambda_s = \lambda_{s_i} = \text{const} \tag{43}$$

$$\lambda_h = \lambda_{s_i}\tan\gamma \tag{44a}$$

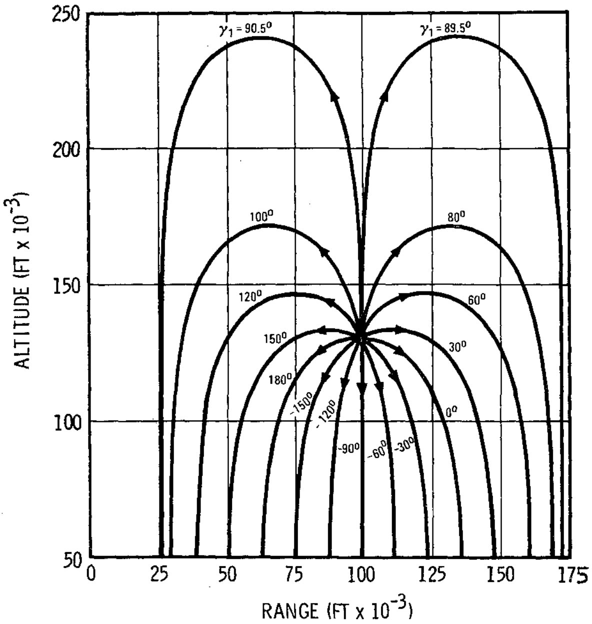

Fig. 1 Family of singular subarcs.

or, if $\gamma_i = -\pi/2$,

$$\lambda_h = \beta c_0\ln V + \text{const} \tag{44b}$$

and

$$\lambda_V V = \text{const} = c_0 \tag{45}$$

This completes the solution on the singular arc for the state variables and Lagrange multipliers.

IV. Solutions on the Nonsingular Subarcs

There are two kinds of nonsingular subarcs, i.e., maximum lift subarcs and minimum lift subarcs. If the assumption of $\max|L/D| \geq 0.5$ is adopted according to Tannas,[8,9]

$$|(g - V^2/R)\cos\gamma| \ll L_{\max}/m \tag{46}$$

on nonsingular subarcs.

Using the aforementioned assumption, the governing equations for the state variables become

$$\dot{V} = -D/m \tag{47}$$

$$\dot{\gamma} = K^*_{LM}Ve^{-\beta h} \tag{48}$$

$$\dot{h} = V\sin\gamma \tag{49}$$

$$\dot{s} = V\cos\gamma \tag{50}$$

where $K^*_{LM} = K_{LM}$ for maximum lift arcs and $K^*_{LM} = -K_{LM}$ for minimum lift arcs. Then Eqs. (47) and (48) give

$$V = V_i\exp\{-K_D/K^*_{LM}(\gamma - \gamma_i)\} \tag{51}$$

or

$$\gamma - \gamma_i = -(K^*_{LM}/K_D)\ln(V/V_i) \tag{52}$$

Similarly, Eqs. (48) and (49) give

$$d\gamma/dh = K^*_{LM}e^{-\beta h}/\sin\gamma \tag{53}$$

then

$$\cos\gamma = (K^*_{LM}/\beta)e^{-\beta h} - C_\gamma \tag{54a}$$

where

$$C_\gamma = (K^*_{LM}/\beta)e^{-\beta h_i} - \cos\gamma_i \tag{54b}$$

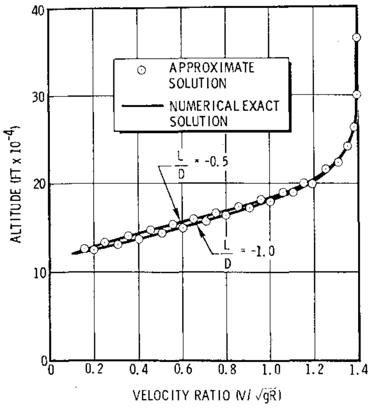

Fig. 2 Minimum lift subarc solutions and numerical solutions.

Finally, Eqs. (48, 50, and 54a) are used to obtain

$$d\gamma/ds = [\beta(\cos\gamma + C_\gamma)/\cos\gamma] \tag{55}$$

or

$$\beta(s - s_i) = \gamma - \gamma_i - C_\gamma \int_{\gamma_i}^{\gamma} \frac{d\gamma}{C_\gamma + \cos\gamma} \tag{56}$$

The integral in Eq. (56) can be evaluated in closed form as follows:

$$\int \frac{d\gamma}{C_\gamma + \cos\gamma} = \frac{2}{(C_\gamma^2 - 1)^{1/2}} \tan^{-1} \frac{(C_\gamma - 1)\tan(\gamma/2)}{(C_\gamma^2 - 1)^{1/2}} \quad \text{if } C_\gamma > 1$$

$$= \tan(\gamma/2) \text{ if } C_\gamma = 1$$

$$= \frac{1}{(1 - C_\gamma^2)^{1/2}} \times \log\left|\frac{(1 - C_\gamma)\tan(\gamma/2) + (1 - C_\gamma^2)^{1/2}}{(1 - C_\gamma)\tan(\gamma/2) - (1 - C_\gamma^2)^{1/2}}\right| \text{ if } C_\gamma < 1 \tag{57}$$

It can be shown that these solutions reduce to the constant density solutions of Ref. 7 if $\beta \to 0$ or $C_\gamma \to \infty$.

The governing equations for the Euler-Lagrange multipliers on the nonsingular arc become

$$\dot{\lambda}_V = \lambda_V D/(mV) \tag{58}$$

$$\dot{\lambda}_\gamma = -\lambda_h V \cos\gamma + \lambda_s V \sin\gamma \tag{59}$$

$$\dot{\lambda}_h = -\beta V[\lambda_h \sin\gamma + \lambda_s \cos\gamma] \tag{60}$$

$$\dot{\lambda}_s = 0 \tag{61}$$

The equation for λ_V is the same as that for the singular arc. Thus,

$$\lambda_V = (V_i/V)\lambda_{V_i} \tag{62}$$

$$\lambda_s = \lambda_{s_i} = \text{const} \tag{63}$$

Equations (48) and (54) give

$$\dot{\gamma} = \pm L_{\max}/mV = K^*_{LM} V e^{-\beta h} = \beta V(C_\gamma + \cos\gamma) \tag{64}$$

The use of Eqs. (60) and (64) yields

$$\frac{d\lambda_h}{d\gamma} = -\frac{1}{C_\gamma + \cos\gamma}\{\lambda_h \sin\gamma + \lambda_{s_i}\cos\gamma\} \tag{65}$$

or

$$\lambda_h = A(C_\gamma + \cos\gamma) - \frac{\lambda_{s_i} C_\gamma \sin\gamma}{C_\gamma^2 - 1} + \frac{\lambda_{s_i}(C_\gamma + \cos\gamma)}{C_\gamma^2 - 1} \int \frac{d\gamma}{C_\gamma + \cos\gamma} \tag{66}$$

where A is an integration constant and the last integral is given in closed form in Eq. (57). Similarly, Eqs. (59) and (64) provide

$$\frac{d\lambda_\gamma}{d\gamma} = -\frac{1}{\beta}\frac{\lambda_h \cos\gamma}{(C_\gamma + \cos\gamma)} + \frac{\lambda_{s_i}\sin\gamma}{(C_\gamma + \cos\gamma)} \tag{67}$$

Substituting Eqs. (66) into (67),

$$\lambda_\gamma = -\frac{A}{\beta}\sin\gamma - \frac{\lambda_s C_\gamma}{\beta(C_\gamma^2 - 1)}(C_\gamma + \cos\gamma) - \frac{\lambda_s \sin\gamma}{\beta(C_\gamma^2 - 1)} \int \frac{d\gamma}{(C_\gamma + \cos\gamma)} + E \tag{68}$$

where E is also an integration constant and the integral is given by Eq. (57). Thus, the Euler-Lagrange multipliers, as well as the state variables, have been obtained in closed form.

The validity of the assumption given in Eq. (46) was tested with the exact numerical solution for some typical cases. The results are given in Figs. 2 and 3 for $\gamma_i = -18°$, $h_i = 360{,}000$ ft, $V_i = 36{,}000$ fps, and $B = 3.2$ psf. It may be seen that our approximate solutions are very close to the exact numerical integration solution. Of course, for the maximum lift subarc, one has to assume that the vehicle will not pull up higher than a certain altitude.

The task of determining the sequence of the subarcs to be employed in the solution of a general problem is very difficult and must be left for future investigations. However, in the following section, three specific examples are chosen to illustrate the questions arising in connecting the singular and nonsingular subarcs.

V. Maximum Final Velocity for Horizontal Flight at a Given Terminal Altitude

The problem of reaching horizontal flight ($\gamma = 0$) at a specific altitude while minimizing the energy loss due to aerodynamic drag (i.e., maximizing the final velocity) is considered under our assumptions for a re-entry glider. A similar problem was solved numerically by Speyer (cf. Ref. 12) and was used as an example by Bryson and Ho.[12]

In order to simplify the subsequent discussions, the numerical results of our analytic solutions are first plotted in Fig. 4 for a simple case for $\gamma_i = -7.5°$, $h_i = 400{,}000$ ft, $V_i = 36{,}000$ fps, and $\gamma_f = 0$, $h_f = 250{,}000$ ft and 275,000 ft, max $|L/D| = 0.5$ and 1.

In this example, if the terminal altitude is sufficiently high, one can easily show that the singular subarc solution cannot be used. The sequence of subarcs is obviously a maximum lift subarc followed by a minimum lift subarc. It is also possible to show that the flight-path angle at the switching point is

$$\cos\gamma_s = (K_{LM}/2\beta)(e^{-\beta h_i} - e^{-\beta h_f}) + \tfrac{1}{2}(\cos\gamma_i + \cos\gamma_f) \tag{69}$$

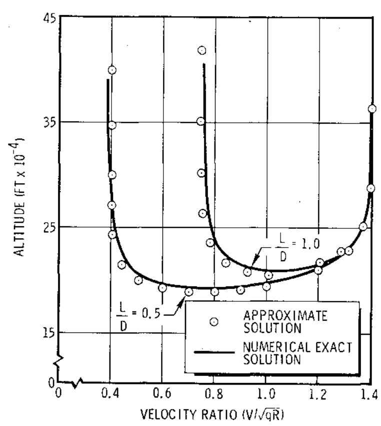

Fig. 3 Maximum lift subarc solutions and numerical solutions.

The other quantities such as h_s, s_s, and V_s can be obtained from Eqs. (51, 54, and 56).

If we want to clarify the questions concerning the number of subarcs, their combination and the location of the switching points, in general, the problem involves more than two subarcs including the singular subarc. As discussed in Ref. 7, it is possible to switch from the maximum lift subarc to the minimum lift subarc or vice versa only at points where $\lambda_\gamma = 0$. In the present example, the final range is not specified. Thus,

$$\lambda_s = 0 \tag{70}$$

Equation (68) then becomes

$$\lambda_\gamma - \lambda_{\gamma_i} = -\frac{1}{\beta}\frac{\lambda_{h_i}(\sin\gamma - \sin\gamma_i)}{C_\gamma + \cos\gamma_i} \tag{71}$$

If we denote the first switching point by $i = 1$, the expression for λ_γ on the second nonsingular subarc becomes

$$\lambda_\gamma = -\lambda_{h_1}(\sin\gamma - \sin\gamma_1)/\beta(C_\gamma + \cos\gamma_1) \tag{72}$$

The earliest possible moment to switch again is at

$$\gamma_2 = \pm\pi - \gamma_1 \tag{73}$$

Geometrically, γ_1 and γ_2 are symmetric with respect to the local vertical. The plus sign is taken on a maximum lift subarc and the minus sign for a minimum lift subarc. Furthermore, if Eq. (70) holds, Eqs. (17, 21, and 23) show that the only possible physical situation for which the Lagrange multipliers on singular subarcs have nontrivial solutions is

$$\gamma = \pm(\pi/2) \tag{74}$$

Using the terminology of Ref. 12, the "possible locus of singular subarcs" is $\gamma = \pm(\pi/2)$. Thus, the only singular subarc which can be used for the case $\lambda_s = 0$ is the vertical descent trajectory.

Another important conclusion resulting from Eq. (52) is that for a sequence of nonsingular subarcs with successive switching points at angles $\gamma_1, \gamma_2, \ldots \gamma_n$, the velocity decreases exponentially with the total sum of the absolute values of the change in flight-path angle on each subarc; i.e.,

$$\frac{V_f}{V_i} = \exp\left\{-\frac{K_D}{K_{LM}}\left[|\gamma_1 - \gamma_i| + |\gamma_2 - \gamma_1| + \ldots + |\gamma_f - \gamma_n|\right]\right\} \tag{75}$$

Thus, if it is desired to reach a certain position by use of one or more nonsingular subarcs, the subarcs should be arranged in such a way that the vehicle turns as little as possible on each subarc. This discussion will help to establish the subsequent mathematical proofs which are necessary for the present problem.

The sequence of subarcs can be determined from geometrical considerations (cf. Fig. 4). If h_{fm} is the lowest altitude that can be reached by an initial maximum lift subarc and if the singular subarcs are disregarded, it is evident by geometrical considerations that the sequence of subarcs is a maximum lift subarc followed by a minimum lift subarc if $h_f > h_{fm}$ and vice versa if $h_f < h_{fm}$.

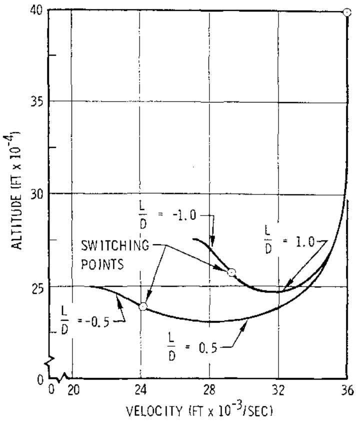

Fig. 4 Maximum velocity for terminal horizontal flight.

However, if the final altitude h_f is sufficiently low so that it cannot be reached by two consecutive nonsingular subarcs or if the flight-path angle assumes a value equal to $-\pi/2$ at certain points on the trajectory, the problem becomes more complicated because the use of the singular subarcs must be considered.

In order to clarify these questions, we first study the following alternatives. In Fig. 5, the vehicle can reach the final horizontal flight position by any of the paths or sequences of subarcs shown in Table 1. The problem is to determine which of the three paths is the optimal one.

Following path I (p_i to p_{11} to p_{1f}), the flight-path angle at the switching point p_{11} is easily obtained;

$$\cos\gamma_{11} = \tfrac{1}{2}(\cos\gamma_i + \cos\gamma_f) + \tfrac{1}{2}(K_{LM}/\beta)(e^{-\beta h_i} - e^{-\beta h_f}) \tag{76}$$

Similarly, the terminal velocity is found to be

$$\ln\frac{V_{1f}}{V_i} = \frac{K_D}{K_{LM}}(2\gamma_{11} - \gamma_i - \gamma_f) = \frac{K_D}{K_{LM}}\left\{-\gamma_i - \gamma_f - \pi + \cos\gamma_i + \cos\gamma_f + \frac{K_{LM}}{\beta}(e^{-\beta h_i} - e^{-\beta h_f}) + 2\left[\frac{\cos^3\gamma_{11}}{2\cdot 3} + \frac{3\cos^5\gamma_{11}}{2\cdot 4\cdot 5} + \ldots\right]\right\} \tag{77}$$

where the $\cos^{-1}\gamma_{11}$-series for $-\pi < \gamma_{11} < 0$ is used. Since we assume $-\pi < \gamma_{11} < -\pi/2$, Eq. (77) gives

$$\ln\frac{V_{1f}}{V_i} < \frac{K_D}{K_{LM}}F(\gamma_i,\gamma_f,h_i,h_f) \tag{78}$$

Table 1

Path	I	II	III
Sequence of subarcs	Min lift ↓ Max lift $\gamma_{11} < -\pi/2$	Min lift ↓ Singular ↓ Max lift	Min lift ↓ Many nonsingular subarcs ↓ Max lift
Switching points		$\gamma_{21} = -\pi/2$	$\gamma_{31}, \gamma_{32}, \ldots \gamma_{3n}$ all close to $-\pi/2$

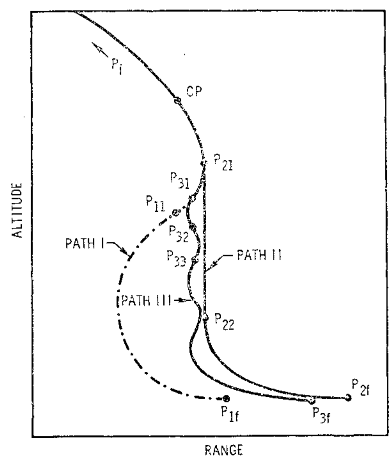

Fig. 5 Sketch of three possible sequences of subarcs.

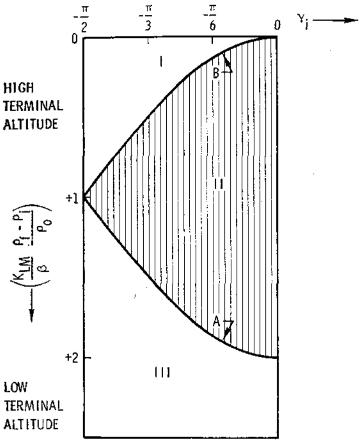

Fig. 6 Boundary conditions and optimal combinations of subarcs for example 1. Area I: maximum lift arc followed by minimum lift arc. Area II: minimum lift arc followed by maximum lift arc. Area III: minimum lift arc followed by singular arc followed by maximum lift arc.

where

$$F = -\gamma_i - \gamma_f - \pi + \cos\gamma_i + \cos\gamma_f + (K_{LM}/\beta)(e^{-\beta h_i} - e^{-\beta h_f}) \quad (79)$$

If path III is followed, the number of nonsingular subarcs needed to reach the terminal horizontal flight position depends on the value of the difference $\gamma_{31} - (-\pi/2)$. In this case, γ_{31} has to be less than $-\pi/2$. Otherwise, one has to fly almost a complete circle before one can switch again because of Eq. (73) which also determines the flight-path angles at all subsequent switching points as follows:

$$\left.\begin{aligned} \gamma_{32} &= -\pi - \gamma_{31} \\ \gamma_{33} &= -\pi - \gamma_{32} = \gamma_{31} \\ \gamma_{34} &= -\pi - \gamma_{33} = \gamma_{32} \\ &\cdots\cdots\cdots\cdots \\ \gamma_{3(2n-1)} &= \gamma_{31} \end{aligned}\right\} \quad (80)$$

Equations (80) and (54) give

$$\cos\gamma_{31} = [1/2(2n-1)]\{\cos\gamma_i + \cos\gamma_f + (K_{LM}/\beta)(e^{-\beta h_i} - e^{-\beta h_f})\} \quad (81)$$

Similarly, the final velocity is obtained as

$$\ln\frac{V_{3f}}{V_i} = \frac{K_D}{K_{LM}}\left\{F + 2(2n-1) \times \left[\frac{\cos^3\gamma_{31}}{2\cdot 3} + \frac{3\cos^5\gamma_{31}}{2\cdot 4\cdot 5} + \ldots\right]\right\} \quad (82)$$

where the arcos-series for $-\pi < \gamma_{31} < 0$ is used. Observing Eq. (81) and the condition $\gamma_{31} < -\pi/2$, we can see that Eq. (82) gives a higher terminal velocity than Eq. (78). Furthermore, Eqs. (80) and (81) show that V_{3f} increases as n increases and that γ_{31} approaches $-\pi/2$ as n approaches infinity.

If path II is followed, the first switching point p_{21} for this case must be located at $\gamma_{21} = -\pi/2$. We can show that the terminal velocity in this case is

$$\ln(V_{2f}/V_i) = (K_D/K_{LM})F \quad (83)$$

Hence, we find from Eqs. (78, 81, and 82) that the optimal path is a minimum lift subarc until $\gamma = -\pi/2$, followed by a vertical singular subarc and finally a maximum lift subarc reaching the terminal horizontal flight position. It is easy to generalize the aforementioned results to the case where the final altitude is sufficiently low so that it can only be reached by more than two nonsingular subarcs or by involving a singular subarc. The optimal path for this case has the same sequence of subarcs. In the foregoing discussion, we have assumed that $\gamma_{11} < -\pi/2$. However, consider the case where the final altitude can be reached by two nonsingular subarcs with the flight-path angle at the switching point

$$\gamma_{11} > -\pi/2 \quad (84)$$

In this case, the possibility of using path II and path III is obviously ruled out. Then the optimal sequence is a minimum lift subarc followed by a maximum lift subarc.

In summary, we conclude that the optimal sequence of subarcs depends essentially on the values of the state variables at initial and terminal conditions (i.e., γ_i, h_i, γ_f, and h_f). Since $\gamma_f = 0$ and h_i and h_f appear always in the combination $[\exp(-\beta h_f) - \exp(-\beta h_i)]$ which is proportional to the difference $(\rho_f - \rho_i)$ in density at the beginning and at the end of the trajectory, the results can be illustrated in a plot representing the $(\rho_f - \rho_i)$, γ_i plane (cf. Fig. 6). Each point on this plane corresponds to a specific set of boundary conditions. The curve "A" represents Eq. (76) with $\gamma_{11} = \gamma_s = -\pi/2$ whereas curve "B" represents the condition $K_{LM}^* = K_{LM}$, $\gamma = \gamma_f = 0$, and $h = h_f$ in Eq. (54) and is equivalent to $h_f = h_{fm}$. The optimal sequence of subarcs can then be immediately visualized from Fig. 6 by entering the point associated with the boundary conditions.

VI. Maximum Terminal Velocity Flight between Two Given Space Points during Re-Entry

As another example, we consider the problem of maximizing the terminal velocity of flight between two given space points during re-entry. In this case, the number of subarcs needed depends on the position of the terminal point relative to the initial point. We will not try to study this problem in general, but discuss only the case where the terminal point can be reached by two subarcs.

We choose the following boundary conditions: V_i = 36,000 fps, $\gamma_i = -30°$, h_i = 360,000 ft, h_f = 115,000 ft, s_f = 274,000 ft, and $\max(L/D) = 3.0$. It is desired to maximize the terminal velocity V_f for any terminal flight-path angle γ_f.

The results are illustrated in Fig. 7. Obviously, the trajectory starts with a minimum lift subarc because the singular subarc at higher altitude is ruled out as discussed in Sec. III. Switching from the nonsingular subarc to the singular subarc is possible anywhere below the critical point. The actual switching point was found in the following manner.

Letting s_1, h_1, and V_1 be the values of the corresponding variables at the switching point, Eqs. (34a) and (35a) give the following relations on the singular subarc:

$$h_1 - h_f = \frac{1}{\beta}\ln\frac{|\cos[\gamma_f - \beta(s_1 - s_f)]|}{|\cos\gamma_f|} \quad (85)$$

and

$$\left.\frac{dh}{ds}\right|_{s=s_1} = \tan[\gamma_f - \beta(s_1 - s_f)] \quad (86)$$

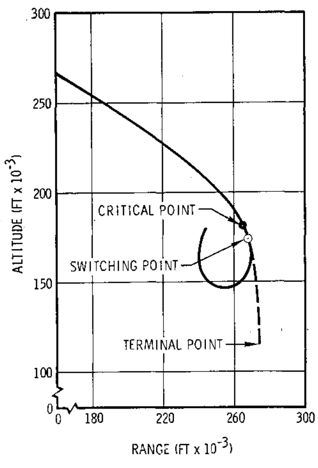

Fig. 8 Maximum velocity to a space point.

This solution represents a family of singular subarcs depending on the parameter γ_f.

Similar conditions can be derived for the nonsingular arc by using Eqs. (54b, 53, 55, and 56);

$$h_1 = -\frac{1}{\beta}\ln\frac{-\beta}{K_{LM}}(C_\gamma + \cos\gamma_1) \quad (87)$$

$$\beta(s_1 - s_i) = \gamma_1 - \gamma_i - C_\gamma \int_{\gamma_i}^{\gamma_1} \frac{d\gamma}{C_\gamma + \cos\gamma} \quad (88)$$

and

$$\left.\frac{dh}{ds}\right|_{s=s_1} = \tan\gamma_1 \quad (89)$$

At the switching point, both the coordinates (h_1,s_1) and the slope $(dh/ds)_{h_1,s_1}$ calculated from the singular subarc solution must agree with the corresponding quantities calculated from the nonsingular subarc solution. Comparison of Eq. (85) with (87), Eq. (86) with (89) and elimination of s by Eq. (88) yields

$$\cos\gamma_f = -\frac{\beta}{K_{LM}} e^{\beta h_f} \cos\gamma_1(C_\gamma + \cos\gamma_1) \quad (90)$$

and

$$\gamma_f = 2\gamma_1 + \beta(s_f - s_i) - \gamma_i - C_\gamma \int_{\gamma_i}^{\gamma_1} \frac{d\gamma}{C_\gamma + \cos\gamma} \quad (91)$$

In the foregoing, we have succeeded in reducing the system into two algebraic equations involving two unknowns (i.e., γ_f and γ_1) only. The quantity γ_f can be plotted against γ_1 and the two curves intersect at the desired values of γ_f and γ_1; h_1 and s_1 can easily be calculated from Eqs. (87) and (88). This scheme was tested numerically for the example given.

This example shows the need of the singular subarc solution since the terminal point cannot be reached by the nonsingular subarc. The lift in the latter exceeds the lift on the singular subarc and causes the vehicle to gain altitude again as shown in Fig. 7. However, it seems that the lift on the singular subarc is controlled in such a way as to enable the vehicle to reach a lower altitude with an almost vertical dive maneuver. A family of singular subarc solutions is shown in Fig. 1.

VII. Maximum Final Velocity Flight to a Terminal Target

The problem considered here is to reach a terminal target point (s_f, h_f) by choosing the initial range s_i so as to maximize the terminal velocity V_f. The problem is mathematically equivalent to prescribing all initial conditions $(V_i, \gamma_i, s_i, h_i)$ and reaching a given altitude h_f with maximum terminal velocity V_f for any s_f and γ_f. Since s_f is not given, Eq. (70) holds and the discussions given at the beginning of Sec. V also apply here.

Although this problem can be discussed in full generality, we restrict the analysis to the case $h_f < h_i$ and $-\pi/2 < \gamma_i < 0$, which covers most cases of practical interest.

It is first necessary to decide which subarc should be used if the final altitude h_f can be reached by either a maximum lift subarc or a minimum lift subarc. This question can be answered by use of the results given in Eq. (51). One can show from the well-known mean value theorem that

$$\cos\gamma_f - \cos\gamma_i = (d/d\gamma)(\cos\gamma)|_{\gamma=\gamma_m}(\gamma_f - \gamma_i) \quad (92)$$

or

$$\gamma_f - \gamma_i = -(\cos\gamma_f - \cos\gamma_i)/\sin\gamma_m \quad (93)$$

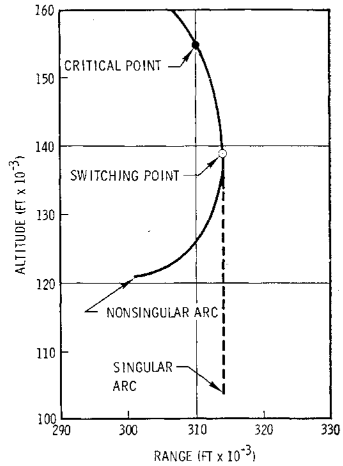

Fig. 8 Maximum velocity to a terminal target.

where

$$\left.\begin{array}{l}\gamma_i < \gamma_m < \gamma_f \text{ for maximum lift subarc}\\ \gamma_f < \gamma_m < \gamma_i \text{ for minimum lift subarc}\end{array}\right\} \quad (94)$$

Equation (54a) also gives

$$\cos\gamma_f - \cos\gamma_i = \pm(K_{LM}/\beta)(e^{-\beta h_f} - e^{-\beta h_i}) \quad (95)$$

The plus sign applies for the maximum lift subarc and the minus sign for the minimum lift subarc. Thus $|\cos\gamma_f - \cos\gamma_i|$ is the same for both subarcs. Since the optimal subarc corresponds to the smaller value of $|\gamma_f - \gamma_i|$, Eq. (93) shows that this is equivalent to finding the subarc which is associated with the larger value of $|\sin\gamma_m|$. The physical situation also indicates that for the maximum lift subarc, the final flight-path angle is less than zero because $h_f < h_i$. Thus the minimum lift subarc is obviously the optimal one.

However, if the final altitude is low enough to be unreachable by a nonsingular subarc without $\gamma = -\pi/2$ at some point along the trajectory, the question of using the singular subarc arises. In this case, $\lambda_s = 0$ and the only possible locus of the singular subarc is $\gamma = \pm(\pi/2)$ along the whole subarc. It is possible to go through the same mathematical argument as given in Sec. V to show that the optimal path is a minimum lift subarc followed by a vertical singular subarc with $\gamma = -\pi/2$.

In summary, one should try to reach the final position by starting with a minimum lift subarc and, if the final altitude cannot be reached before $\gamma = -\pi/2$, one should switch to the singular subarc at $\gamma = \gamma_s = -\pi/2$ and descend vertically until the final altitude h_f is reached. A numerical example is given in Fig. 8 for $V_i = 36{,}000$ fps, $\gamma_i = -30°$, $h_i = 360{,}000$ ft, max $(L/D) = 1.0$ and $B = 3.2$ psf. Both trajectories, consisting of the minimum lift subarc only and the minimum lift subarc followed by a singular subarc, end at the same terminal velocity $V_f = 3000$ fps. It can be seen that the trajectory with the singular subarc reaches a lower altitude which agrees with our theoretical discussions.

VIII. Conclusion and Discussion

In the foregoing analysis, the gravity term in the velocity equation was neglected in comparison with the drag force. The effect of such a term on the calculated lifting force for the singular subarc can be estimated. If the gravity term is not neglected,

$$\dot{\gamma} = -\beta V \cos\gamma(1 + 2g/\beta V^2) \quad (96)$$

The ratio of the neglected term to the calculated lift on the singular subarc is about 0.03 if we restrict ourself to the velocity range above 2000 fps. Thus, in the range of hypersonic flight, the neglected term is of higher order.

Another distinct feature of the lift on the singular subarc is the dominating effect of the term (i.e., $-mV^2\beta \cos\gamma$) produced by the variation of the atmospheric density on the lifting force which is totally absent in the constant density case discussed by Pottsepp and Shi.[7] It was shown earlier that boundedness of the control variable precludes application of singular subarcs above a certain altitude.

It can be shown that the singular subarc appearing in the present problem belongs to a class of singular subarcs investigated by Kelly, Kopp,[13] and Bryson.[12] By making use of special variations, Kelly and Kopp[13] developed an additional convexity condition which is to be satisfied by the singular subarc. For the present problem, the additional condition reduces to

$$\frac{\partial}{\partial\alpha}\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial\alpha}\right) = \frac{1}{m^2V}\left(\frac{\partial L}{\partial\alpha}\right)^2(\lambda_x\cos\gamma + \lambda_y\sin\gamma)$$

$$= \frac{(\lambda_v V)D}{m^3V^3}\left(\frac{\partial L}{\partial\alpha}\right)^2 > 0 \qquad (97)$$

which is nontrivially satisfied on the singular arc because $\lambda_v V = V(T) > 0$. Furthermore, the necessary conditions at the corners where the singular subarcs join the nonsingular subarc discussed in Ref. 13 are also satisfied.

The assumption of constant drag coefficient made by various previous investigators (cf. Refs. 1–4) is a reasonable approximation for a blunt hypersonic vehicle[7] and is an excellent approximation for a class of variable geometry vehicles considered by Chapman.[2] The lifting body (model B) discussed by Boylan and Potter[14] would fall into this category if one restricts $-10° < \alpha < -10°$. Furthermore, the Gemini type body (model D) of the same reference would also fall into this category if the modified Newtonian results are used. Finally, the present problem, parallel to Ref. 7, presents a nontrivial, instructive example where the singular subarc is needed.

References

[1] Allen, H. J. and Eggers, A. J., Jr., "A Study of the Motion and Aerodynamic Heating of Ballistic Missile Entering the Earth's Atmosphere at High Supersonic Speeds," Rept., 1958, NACA; *AIAA Selected Reprint Series, Vol. 1, Atmosphere Entry*, AIAA, New York, 1967, pp. 12–24.

[2] Chapman, D. R., "On the Corridor and Associated Trajectories Accuracy for Entry of Manned Spacecraft into Planetary Atmospheres," *Proceedings of the Xth International Astronautical Congress*, Springer-Verlag, Vienna, 1960, pp. 254–267.

[3] Loh, W. H. T., *Dynamics and Thermodynamics of Planetary Re-Entry*, Prentice-Hall, Englewood Cliffs, N.J., 1963.

[4] Citron, S. J. and Meir, T. C., "An Analytic Solution for Entry into Planetary Atmospheres," *AIAA Journal*, Vol. 3, No. 3, March 1965, pp. 470–475.

[5] Miele, A., *Optimization Techniques With Applications to Aerospace Systems*, edited by G. Leitmann, Academic Press, New York, 1962, Chaps. III and IV.

[6] Miele, A., "On The Flight Path of A Hypersonic Glider," Rept. A-59-2, 1959, Purdue Univ., Lafayette, Ind.

[7] Pottsepp, L. and Shi, Y. Y., "Optimal Lift Control of a Hypersonic Glider," *AIAA Journal*, Vol. 6, No. 6, June 1968, pp. 1169–1171.

[8] Tannas, L. E., Jr., "Re-Entry Guidance through Closed Form Equations," *AIAA Journal*, Vol. 5, No. 6, June 1967, pp. 1102–1109.

[9] Tannas, L. E., Jr. and Perkins, T. R., "Simulation Evaluation of Closed Form Re-Entry Guidance," AIAA Paper 67-597, Huntsville, Ala., 1967.

[10] Breakwell, J. V., "The Optimization of Trajectories," *SIAM Journal*, Vol. 7, No. 2, June 1959, pp. 215–247.

[11] Leitmann, G., *Optimization Techniques With Applications to Aerospace Systems*, edited by G. Leitmann, Academic Press, New York, 1962, Chap. V, pp. 170–205.

[12] Bryson, A. E. and Ho, Y. C., *Optimization, Estimations and Control*, Blaisdell, Waltham, Mass., to be published.

[13] Kelley, H. J., Kopp, R. E., and Moyer, H. E., *Topics in Optimizations*, edited by G. Leitmann, Academic Press, New York, 1967, Chap. III.

[14] Boylan, D. E. and Potter, J. L., "Aerodynamics of Typical Lifting Bodies Under Conditions Simulating Very High Altitudes," *AIAA Journal*, Vol. 5, No. 2, Feb. 1967, pp. 226–232.